Haiman Math 1A—Calculus Fall, 2006

Second Midterm Exam Solutions

Name ____________________ Student ID ____________________

Discussion Section (Time and GSI) ____________________

You may use one sheet of notes. No other notes, books or calculators allowed. There are 8 questions, on front and back. Write answers on the exam and turn in only this paper. Show enough work so that we can see how you arrived at your answers.

1. Find the equation of the tangent line to the hyperbola $x^2 + 2xy - y^2 + x = 9$ at the point $(2, 1)$.

Differentiate implicitly with respect to x, to get

$$2x + 2y + 2xy' - 2yy' + 1 = 0.$$

Set $x = 2$, $y = 1$, and solve for $y' = -7/2$. The equation of the tangent line is then

$$y = (-7/2)(x - 2) + 1 = 8 - 7x/2.$$

2. Differentiate the function $(\ln x)^{\cos x}$.

Using logarithmic differentiation,

$$f'(x) = f(x)\frac{d}{dx}\ln f(x) = (\ln x)^{\cos x}\big(\frac{\cos x}{x \ln x} - (\sin x)(\ln\ln x)\big)$$

3. The surface area of a melting spherical ball of ice decreases at a rate of 1 cm^2/min. How fast is the volume decreasing when the radius of the ball is 10 cm? For your information, the surface area and volume of a sphere of radius r are given by $A = 4\pi r^2$ and $V = 4\pi r^3/3$.

$dA/dt = 8\pi r\, dr/dt$, $dV/dt = 4\pi r^2\, dr/dt = (r/2)\, dA/dt$. Substitute $r = 10$, $dA/dt = -1$ to get $dV/dt = -5$, so the volume is decreasing at a rate of 5 cm^3/min.

4. Use a linear approximation or differentials to estimate the value of $\sqrt[4]{1.2}$

Using differentials, with $x = 1$, $dx = .2$, $dy/dx = x^{-3/4}/4 = 1/4$, get $dy = .05$, giving $\sqrt[4]{1.2} \approx 1.05$.

5. Use the Mean Value Theorem to determine whether the estimate for $\sqrt[4]{1.2}$ obtained by linear approximation is greater or less than the actual value.

The linear approximation is $1.05 = 1 + f'(1)(1.2-1) = 1 + (1/4)(1.2-1)$. By MVT, the exact value is $\sqrt[4]{1.2} = 1 + f'(c)(1.2-1)$ for some $c \in (1, 1.2)$. Now $f'(c) = (1/4)c^{-3/4} < 1/4$ for $c > 1$, therefore $\sqrt[4]{1.2} < 1.05$.

6. Find the minimum and maximum values, and any local minima and maxima in the interior of the interval, of the function $f(x) = x^2 + x - 2|x|$ on $[-2, 2]$.

The derivative is

$$f'(x) = \begin{cases} 2x - 1 & x > 0, \\ 2x + 3 & x < 0 \end{cases}$$

and undefined at $x = 0$. This gives three critical points at $x = 1/2$, $x = 0$ and $x = -3/2$. Evaluate f at the critical points and endpoints:
$f(-2) = -2$.
$f(-3/2) = -9/4$ is the minimum (hence also a local minimum).
$f(0) = 0$ is a local maximum by the 1st derivative test.
$f(1/2) = -1/4$ is a local minimum by either the 1st or 2nd derivative test.
$f(2) = 2$ is the maximum.

7. Find the limit

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

Express as a fraction and use L'Hospital's rule twice:

$$\begin{aligned} \lim_{x \to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} &= \lim_{x \to 1} \frac{1 - 1/x}{1 - 1/x + \ln x} \\ &= \lim_{x \to 1} \frac{1/x^2}{1/x^2 + 1/x} = \frac{1}{2}. \end{aligned}$$

8. Find the area of the largest rectangle in the first quadrant with one side on the x-axis, one side on the y-axis, and the opposite vertex on the parabola $y = 27 - x^2$.

We are to maximize $A(x) = xy = 27x - x^3$ on $[0, \sqrt{27}]$. Since $A'(x) = 27 - 3x^2$, we see that $x = 3$ is a critical point. The area is zero at the endpoints $x = 0$, $x = \sqrt{27}$, so the maximum is $A(3) = 54$.